ATKINSON SCIENCE

THEORY GUIDE

Admittance Method

5 Dynamic Thermal Parameters

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Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

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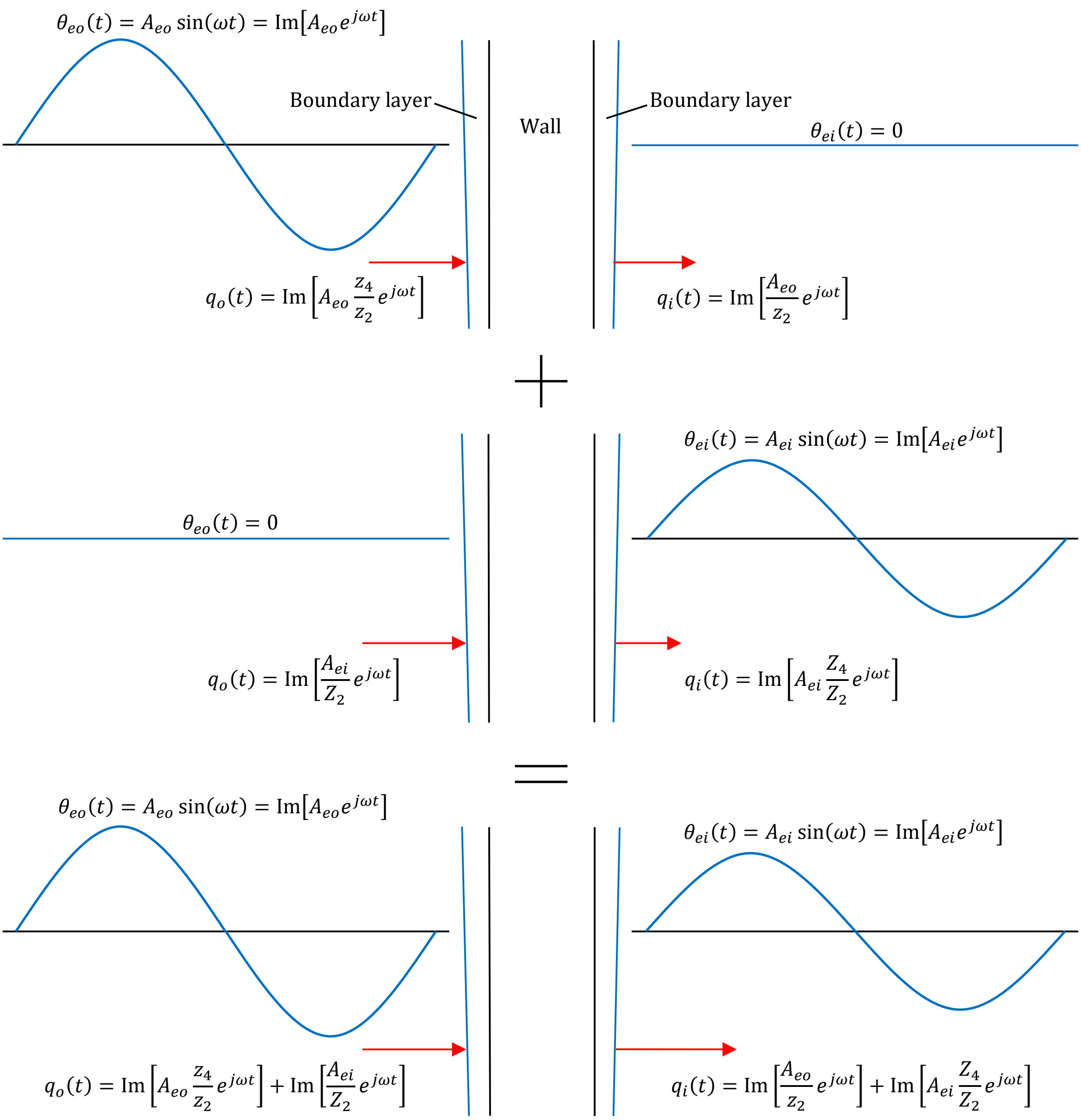
1 Introduction

In the previous report in this series, Ref. [1], we derived the complex transmission matrix and the inverse complex transmission matrix for a composite wall with a boundary layer on each side. We showed how the two matrices can be used to determine the net heat flux through each side of the wall when the sol-air temperature and the environmental temperature are fluctuating sinusoidally. Figure 1 summarises the calculation method. We use the superposition principle to add the heat fluxes calculated when the sol-air temperature is fluctuating sinusoidally and the environmental temperature is constant to the heat fluxes calculated when environmental temperature is fluctuating sinusoidally and the sol-air temperature is constant. z_2 and z_4 are elements of the transmission matrix and Z_2 and Z_4 are elements of the inverse transmission matrix.

Figure 1 Superposition principle used to calculate the heat fluxes through a wall



In this report we shall review how the two matrices are used calculate the heat fluxes on the two sides of a composite wall with boundary layers. Then we shall introduce some *dynamic thermal parameters* associated with the matrices.

2 Transmission matrix

To apply the transmission matrix to the heating of buildings, we assume that the variation in the sol-air temperature $\theta_{eo}(t)$ on the outdoor side of the wall is sinusoidal and the environmental temperature on the indoor side $\theta_{ei}(t)$ is zero:

$$\begin{bmatrix} A_{eo} \\ Q_o \end{bmatrix} = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \begin{bmatrix} 0 \\ Q_i \end{bmatrix} \quad (2.1)$$

The time-varying sol-air temperature is given by

$$\theta_{eo}(t) = A_{eo}\sin(\omega t) = \mathrm{Im}\left(A_{eo}e^{j\omega t}\right) \quad (2.2)$$

where A_{eo} is the amplitude of the sol-air temperature and Im means "the imaginary part of". Q_o and Q_i are complex constants. In (2.1) A_{eo} is used as a reference temperature and the phases of all the other quantities are determined with respect to the temperature $\theta_{eo}(t)$.

From (2.1) we obtain:

$$A_{eo} = z_2 Q_i \quad (2.3)$$

and

$$Q_o = z_4 Q_i \quad (2.4)$$

From these two equations we obtain

$$Q_i = \frac{A_{eo}}{z_2} \quad (2.5)$$

and

$$Q_o = A_{eo}\frac{z_4}{z_2} \quad (2.6)$$

The instantaneous heat flux through the inside surface of the wall is

$$q_i(t) = \mathrm{Im}\left[Q_i e^{j\omega t}\right] = \mathrm{Im}\left[\frac{A_{eo}}{z_2}e^{j\omega t}\right] \quad (2.7)$$

and the instantaneous heat flux through the outside surface is

$$q_o(t) = \mathrm{Im}\left[Q_o e^{j\omega t}\right] = \mathrm{Im}\left[A_{eo}\frac{z_4}{z_2}e^{j\omega t}\right] \quad (2.8)$$

The transmission matrix for a planar structure with n layers is

$$\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} = \begin{bmatrix} 1 & 1/h_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh M_1 & \dfrac{\sinh M_1}{N_1} \\ N_1 \sinh M_1 & \cosh M_1 \end{bmatrix} \begin{bmatrix} \cosh M_2 & \dfrac{\sinh M_2}{N_2} \\ N_2 \sinh M_2 & \cosh M_2 \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} \cosh M_{n-1} & \dfrac{\sinh M_{n-1}}{N_{n-1}} \\ N_{n-1} \sinh M_{n-1} & \cosh M_{n-1} \end{bmatrix} \begin{bmatrix} \cosh M_n & \dfrac{\sinh M_n}{N_n} \\ N_n \sinh M_n & \cosh M_n \end{bmatrix} \begin{bmatrix} 1 & 1/h_i \\ 0 & 1 \end{bmatrix} \qquad (2.9)$$

where h_o is the convection heat transfer coefficient on the outdoor face of the wall and h_i is the coefficient on the indoor face. The complex constants M_i and N_i are

$$M_i = l_i \sqrt{j\omega/\alpha_i} \qquad (2.10)$$

and

$$N_i = k_i \sqrt{j\omega/\alpha_i} \qquad (2.11)$$

In M_i and N_i, j is the imaginary constant $\sqrt{-1}$, ω [rad s^{-1}] is the angular frequency of the sinusoidal variation in the sol-air temperature, k_i [W m^{-1} K^{-1}] is the thermal conductivity of the ith layer, α_i [m^2 s^{-1}] is the thermal diffusivity of the ith layer, and l_i [m] is the thickness of the ith layer.

The angular frequency ω is

$$\omega = \frac{2\pi}{T} \qquad (2.12)$$

where T is the period of the fluctuations in sol-air temperature. T is 24 hours (= 86400 s), so

$$\omega = \frac{2\pi}{86400} = 7.2722 \times 10^{-5} \text{ rad s}^{-1} \qquad (2.13)$$

The thermal diffusivity of the ith layer α_i is given by

$$\alpha_i = \frac{k_i}{\rho_i C_i} \qquad (2.14)$$

where ρ_i [kg m^{-3}] and C_i [J kg^{-1} K^{-1}] are the density and the specific heat capacity of the ith layer, respectively.

3 Inverse transmission matrix

In the previous report in this series, Ref. [1], we derived the inverse transmission matrix:

$$\begin{bmatrix} A_{ei} \\ Q_i \end{bmatrix} = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} 0 \\ Q_o \end{bmatrix} \quad (3.1)$$

The time-varying environmental temperature is given by

$$\theta_{ei}(t) = A_{ei} \sin(\omega t) = \mathrm{Im}\left(A_{ei} e^{j\omega t}\right) \quad (3.2)$$

where A_{ei} is the amplitude of the environmental temperature and Im means "the imaginary part of". Q_i and Q_o are complex constants. In (3.1) A_{ei} is used as a reference temperature and the phases of all the other quantities are determined with respect to the temperature $\theta_{ei}(t)$.

From (3.1) we obtain:

$$A_{ei} = Z_2 Q_o$$

and

$$Q_i = Z_4 Q_o$$

From these two equations we obtain

$$Q_o = \frac{A_{ei}}{Z_2}$$

and

$$Q_i = A_{ei} \frac{Z_4}{Z_2}$$

The instantaneous heat flux through the inside surface of the wall is

$$q_i(t) = \mathrm{Im}\left[Q_i e^{j\omega t}\right] = \mathrm{Im}\left[A_{ei} \frac{Z_4}{Z_2} e^{j\omega t}\right] \quad (3.3)$$

and the instantaneous heat flux through the outside surface is

$$q_o(t) = \mathrm{Im}\left[Q_o e^{j\omega t}\right] = \mathrm{Im}\left[\frac{A_{ei}}{Z_2} e^{j\omega t}\right] \quad (3.4)$$

The inverse transmission matrix for a planar structure with n layers is

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} = \begin{bmatrix} 1 & -1/h_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh M_n & -\dfrac{\sinh M_n}{N_n} \\ -N_n \sinh M_n & \cosh M_n \end{bmatrix} \begin{bmatrix} \cosh M_{n-1} & -\dfrac{\sinh M_{n-1}}{N_n} \\ -N_n \sinh M_n & \cosh M_n \end{bmatrix} \cdots$$

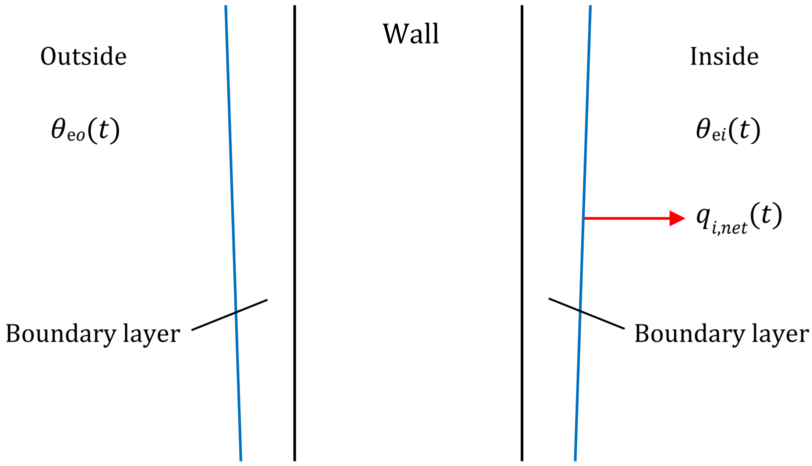
$$\cdots \begin{bmatrix} \cosh M_2 & -\dfrac{\sinh M_2}{N_2} \\ -N_2 \sinh M_2 & \cosh M_2 \end{bmatrix} \begin{bmatrix} \cosh M_1 & -\dfrac{\sinh M_1}{N_1} \\ -N_1 \sinh M_1 & \cosh M_1 \end{bmatrix} \begin{bmatrix} 1 & -1/h_o \\ 0 & 1 \end{bmatrix} \quad (3.5)$$

where the complex constants M_i and N_i are defined by (2.10) and (2.11), respectively.

4 Net heat flux through the building envelope

We now wish to determine the net heat flux $q_{i,net}$ in Figure 2 in response to sinusoidal variations in the sol-air temperature $\theta_{eo}(t)$ and the environmental temperature $\theta_{ei}(t)$.

Figure 2 Building envelope



The contribution to $q_{i,net}(t)$ from the sol-air temperature $\theta_{eo}(t)$ is given by (2.7):

$$q_i(t) = \mathrm{Im}\left[Q_i e^{j\omega t}\right] = \mathrm{Im}\left[\frac{A_{eo}}{z_2} e^{j\omega t}\right] \qquad (2.7)$$

and the contribution to $q_{i,net}$ from the environmental temperature $\theta_{ei}(t)$ is given by (3.3):

$$q_i(t) = \mathrm{Im}\left[Q_i e^{j\omega t}\right] = \mathrm{Im}\left[A_{ei} \frac{Z_4}{Z_2} e^{j\omega t}\right] \qquad (3.3)$$

By virtue of the superposition principle, we can add the two contributions together to obtain the net heat flux.

In the following sections we shall introduce some *dynamic thermal parameters*. The most important of these parameters are tabulated in Ref. [2] for different types of planar composite wall. Two parameters are of particular importance: the *decrement factor* and the *thermal admittance*. The first of these can be used to evaluate (2.7) and the second can be used to evaluate (3.3). Consequently, we can evaluate $q_{i,net}$. In addition, there is a third parameter, the *surface factor*, that can be used determine the effect of solar gain on the temperature in a building. We shall present worked examples to show how each dynamic thermal parameter is calculated.

5 Periodic thermal transmittance

We can rewrite (2.7) as:

$$q_i(t) = \mathrm{Im}\left[\frac{A_{eo}}{z_2} e^{j\omega t}\right] = A_{eo}\mathrm{Im}\left[X_c e^{j\omega t}\right] \quad (5.1)$$

where the complex constant X_c is

$$X_c = \frac{1}{z_2} \quad (5.2)$$

The complex constant X_c can be written in the modulus-argument (amplitude-phase) form:

$$X_c = |X_c| e^{j\mathrm{Arg}(X_c)} \quad (5.3)$$

where

$$|X_c| = \sqrt{\mathrm{Re}(X_c)^2 + \mathrm{Im}(X_c)^2} \quad (5.4)$$

is the modulus or amplitude of X_c and

$$\mathrm{Arg}(X_c) = \mathrm{atan}\left[\frac{\mathrm{Im}(X_c)}{\mathrm{Re}(X_c)}\right] \quad (5.5)$$

is the argument or phase of X_c.

Substituting (5.3) into (5.1) gives

$$q_i(t) = A_{eo}\mathrm{Im}\left[|X_c| e^{j[\omega t + \mathrm{Arg}(X_c)]}\right] \quad (5.6)$$

The terms $|X_c|$ and $\mathrm{Arg}(X_c)$ in (5.6) have special names. $|Xc|$ is known as the *periodic thermal transmittance* and is denoted by X. $\mathrm{Arg}(X_c)$ is known as the *periodic thermal transmittance time lag* and is denoted by λ. The parameter λ is a time *lag*, so it will be negative. In tables of dynamic thermal parameters λ is given as a positive number, so we must place a negative sign in front of it in (5.6). Thus

$$q_i(t) = A_{eo}\mathrm{Im}\left[X e^{j[\omega t - \lambda]}\right] = X A_{eo} \sin(\omega t - \lambda) \quad (5.7)$$

6 Example 1

(a) For the composite wall with boundary layers in Ref. [1], determine the periodic thermal transmittance and the periodic thermal transmittance time lag. (b) The sol-air temperature is cyclic with a mean of 0°C and an amplitude of 5°C. The environmental temperature is constant at 0°C. Plot the sol-air temperature $\theta_{eo}(t)$ and the inside heat flux $q_i(t)$ against time.

(a) For the composite wall in Ref. [1], the transmittance matrix is

$$\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} = \begin{bmatrix} (-6.31935 + j1.46011) & (-4.58586 + j5.36354) \\ (-47.0447 - j15.6345) & (-51.4265 + j16.7011) \end{bmatrix} \qquad (6.1)$$

and the inverse transmission matrix is

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} = \begin{bmatrix} (-51.4321 + j16.6913) & (4.58715 - j5.36282) \\ (47.0435 + j15.6448) & (-6.31991 + j1.45887) \end{bmatrix} \qquad (6.2)$$

The complex constant X_c [W m^{-2} K^{-1}] required in the periodic thermal transmittance is defined by Eq. (5.2):

$$X_c = \frac{1}{z_2}$$

Substituting z_2 from (6.1) gives

$$X_c = \frac{1}{(-4.58586 + j5.36354)}$$

$$= \frac{(-4.58586 - j5.36354)}{(-4.58586 + j5.36354)(-4.58586 - j5.36354)}$$

$$= \frac{-4.58586 - j5.36354}{4.58586^2 - j^2 5.36354^2}$$

$$= \frac{-4.58586 - j5.36354}{49.7977}$$

$$= -0.09209 - j0.10771$$

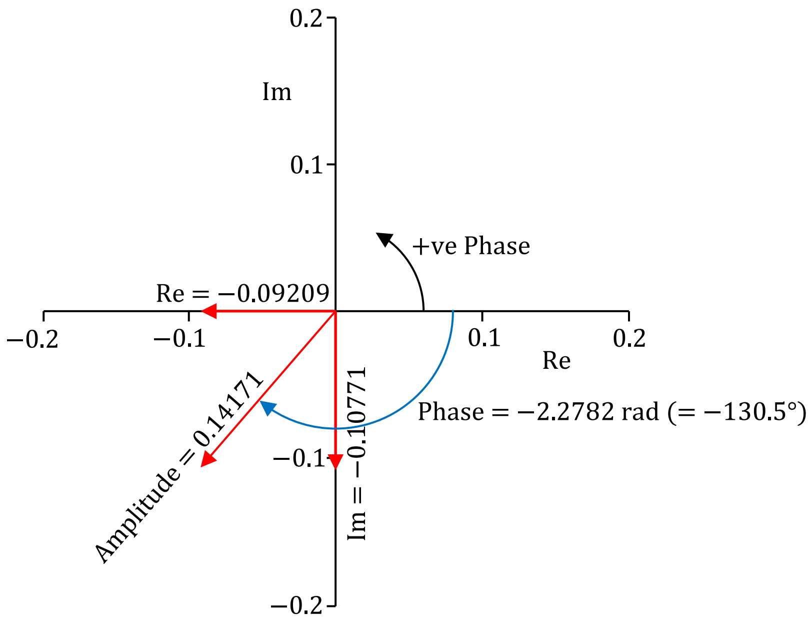
The complex constant X_c can be represented in the complex plane as shown in Figure 3. The amplitude of X_c is

$$\text{Amplitude} = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{(-0.09209)^2 + (-0.10771)^2} = 0.14171$$

The phase of a complex number is measured anticlockwise from the positive Real axis. We know the heat flux variation on the inside surface of the wall will lag the temperature variation on the outside surface, so the phase of the heat flux variation will be negative relative to the temperature variation. Measuring the phase in the clockwise (negative) direction from the positive Real axis gives

$$\text{Phase} = -2.2782 \text{ rad } (= -130.5°)$$

Figure 3 Amplitude and phase of X_c



We can now write X_c as

$$X_c = 0.14171[\cos(-2.2782) + j\sin(-2.2782)]$$

$$= 0.14171e^{-j2.2782}$$

Thus the amplitude X is 0.14171 [W m^{-2} K^{-1}] and the phase is −2.2782 rad and represents a time *lag*. In tables the periodic thermal transmittance time lag λ is given as a positive value, so it will be 2.2782 rad.

In terms of hours, the periodic thermal transmittance time lag λ is

$$\lambda = \frac{24}{2\pi} \times 2.2782 = 8.7021 \text{ hr (8 hr 42 min)}$$

The indoor heat flux is given by (5.7):

$$q_i(t) = XA_{eo}\sin(\omega t - \lambda)$$

$$= 0.14171 \times 5 \times \sin(\omega t - \lambda)$$

$$= 0.70855\sin(\omega t - \lambda)\ \ [\mathrm{W\,m^{-2}}]$$

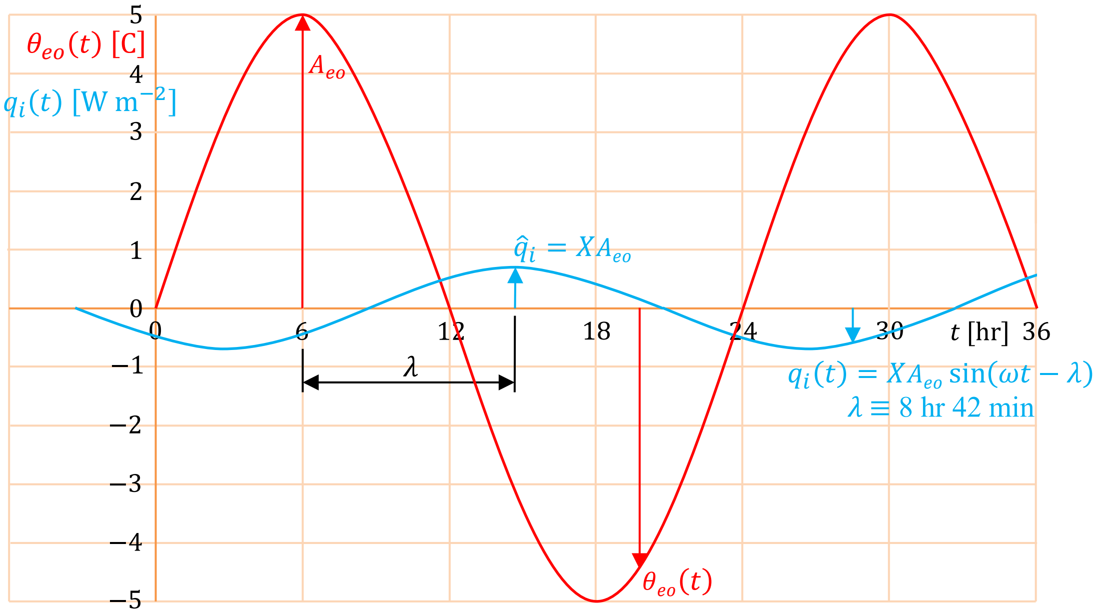
(b) Figure 4 shows the sol-air temperature $\theta_{eo}(t)$ and the heat flux $q_i(t)$ against time. $A_{eo} = 5°\text{C}$ is the amplitude of the sol-air temperature and $\hat{q}_i = XA_{eo} = 0.70855 \text{ W m}^{-2}$ is the amplitude of the heat flux.

For thin structures with very little thermal capacity λ tends to zero and (5.7) becomes

$$q_i(t) = XA_{eo}\sin(\omega t) = X\theta_{eo}(t)$$

At any instant $q_i(t)$ is directly proportional to $\theta_{eo}(t)$, as in steady-state thermal conduction. It follows that the constant of proportionality X must be equal to the steady thermal transmittance U.

Figure 4 Illustration of the periodic thermal transmittance



7 Decrement factor

The decrement factor is similar in concept to the periodic thermal transmittance. We can rewrite (2.7) as:

$$q_i(t) = \mathrm{Im}\left[\frac{A_{eo}}{z_2}e^{j\omega t}\right] = A_{eo}\mathrm{Im}\left[X_c e^{j\omega t}\right] = A_{eo}U\mathrm{Im}\left[\frac{X_c}{U}e^{j\omega t}\right] \qquad (7.1)$$

where U is the steady thermal transmittance. The complex constant X_c is just the same as that in the periodic thermal transmittance:

$$X_c = \frac{1}{z_2} \qquad (7.2)$$

and we can write X_c in modulus-argument form just as before:

$$X_c = |X_c|e^{j\mathrm{Arg}(X_c)} \qquad (7.3)$$

where

$$|X_c| = \sqrt{\mathrm{Re}(X_c)^2 + \mathrm{Im}(X_c)^2} \qquad (7.4)$$

and

$$\mathrm{Arg}(X_c) = \mathrm{atan}\left[\frac{\mathrm{Im}(X_c)}{\mathrm{Re}(X_c)}\right] \qquad (7.5)$$

Substituting (7.3) into (7.1) gives

$$q_i(t) = A_{eo}U\mathrm{Im}\left[\frac{|X_c|}{U}e^{j[\omega t + \mathrm{Arg}(X_c)]}\right] \qquad (7.6)$$

The terms $|X_c|/U$ and $\mathrm{Arg}(X_c)$ in (7.6) have special names. $|Xc|/U$ is known as the *decrement factor* and is denoted by f. $\mathrm{Arg}(X_c)$ is known as the *decrement factor time lag* and is denoted by ϕ. (Note that the decrement factor time lag ϕ is the same as the periodic thermal transmittance time lag λ.)

The parameter ϕ is a time *lag* (the peak in $q_i(t)$ lags the peak in $\theta_{eo}(t)$), so ϕ should always be negative. In tables of dynamic thermal parameters, the decrement factor time lag is given as a positive number, so we must place a negative sign in front of it in (7.6). Thus

$$q_i(t) = A_{eo}U\mathrm{Im}\left[fe^{j[\omega t - \phi]}\right] = UfA_{eo}\sin(\omega t - \phi) \qquad (7.7)$$

8 Example 2

For the composite wall with boundary layers in Ref. [1], determine the decrement factor and the decrement factor time lag.

We calculated the complex constant X_c required in the decrement factor in Example 1:

$$X_c[\mathrm{W\ m^{-2}K^{-1}}] = |X_c|e^{j\mathrm{Arg}(X_c)} = 0.14171e^{-j2.2782}$$

We calculated steady thermal transmittance required in the decrement factor in Ref. [1]:

$$U = 0.58631\ \mathrm{W\ m^{-2}\ K^{-1}}$$

The decrement factor f [ ] is therefore

$$f = \frac{|X_c|}{U} = \frac{0.14171}{0.58631} = 0.24170$$

The decrement factor time lag ϕ is the same as the periodic thermal transmittance time lag λ. In terms of hours ϕ is

$$\phi = \lambda = 8.7021\ \mathrm{hr}\ (8\ \mathrm{hr}\ 42\ \mathrm{min})$$

For thin structures with very little thermal capacity λ tends to zero and (7.7) becomes

$$q_i(t) = UfA_{eo}\sin(\omega t) = Uf\theta_{eo}(t)$$

At any instant $q_i(t)$ is directly proportional to $\theta_{eo}(t)$, as in steady-state thermal conduction. It follows that the constant of proportionality Uf must be equal to the steady thermal transmittance U and consequently f must be equal to 1.

9 Thermal admittance

The instantaneous heat flux $q_i(t)$ at the inside of a composite wall due to a sinusoidally varying environmental temperature $\theta_{ei}(t)$ with amplitude A_{ei} and angular frequency ω is given by (3.3):

$$q_i(t) = \mathrm{Im}\left[A_{ei}\frac{Z_4}{Z_2}e^{j\omega t}\right] \quad (3.3)$$

where Z_2 and Z_4 are elements of the inverse transmission matrix. We define the complex constant Y_c as

$$Y_c = -\frac{Z_4}{Z_2} \quad (9.1)$$

The negative sign has the effect of subtracting π radians from the phase of Z_4/Z_2. For diurnal temperature variations, π radians is equivalent to 12 hours.

If we substitute Y_c rather than Z_4/Z_2 into (3.3) then we will find that the phase lead of $q_i(t)$ in relation to $\theta_{ei}(t)$ is reduced by 12 hours. The phase lead will become the time difference between the peak *negative* heat flux and the peak positive temperature, rather than the time difference between the peak *positive* heat flux and the peak positive temperature. Substituting Y_c rather than Z_4/Z_2 into (3.3) appears logical because a positive environmental temperature is associated with a negative heat flux (a heat flux in the negative x direction).

The complex constant Y_c can be written in the modulus-argument form:

$$Y_c = |Y_c|e^{j\mathrm{Arg}(Y_c)} \quad (9.2)$$

The terms $|Y_c|$ and $\mathrm{Arg}(Y_c)$ in (9.2) have special names. $|Y_c|$ is known as the *thermal admittance* and is denoted by Y. $\mathrm{Arg}(Y_c)$ is known as the *thermal admittance time lead* and is denoted by φ. Substituting (9.2) into (3.3) gives

$$q_i(t) = \mathrm{Im}\left[A_{ei}Y_c e^{j\omega t}\right] = \mathrm{Im}\left[A_{ei}Y e^{j[\omega t+\varphi]}\right] = A_{ei}Y\sin(\omega t + \varphi) \quad (9.3)$$

Note that this heat flux makes a *negative* contribution to the net heat flux $q_{i,net}(t)$ because the time lead is the time between the peak negative heat flux and the peak environmental temperature.

10 Example 3

For the composite wall with boundary layers in Ref. [1], determine the thermal admittance and the thermal admittance time lead.

The complex constant Y_c required in the thermal admittance is defined by (9.1):

$$Y_c = -\frac{Z_4}{Z_2}$$

The elements of the inverse transmission matrix Z_2 and Z_4 were given in Example 1. Substituting gives

$$Y_c = -\frac{(-6.31991 + j1.45887)}{(4.58715 - j5.36282)}$$

$$= -\frac{(-6.31991 + j1.45887)(4.58715 + j5.36282)}{(4.58715 - j5.36282)(4.58715 + j5.36282)}$$

$$= -\frac{-28.99037 - j33.89254 + j6.69205 + j^2 7.82366}{4.58715^2 + 5.36282^2}$$

$$= -\frac{-36.81403 - j27.20049}{49.80178}$$

$$= 0.73921 + j0.54617$$

$$= 0.91909[\cos(0.63633) + j\sin(0.63633)]$$

$$= 0.91909e^{j0.63633}$$

Thus the modulus $|Yc|$ is 0.91909 [W m^{-2} K^{-1}] and the argument φ is 0.63633 rad. Note that φ is positive and represents a time *lead*.

The admittance Y [W m^{-2} K^{-1}] is the modulus of Y_c so Y is 0.91909 W m^{-2} K^{-1}.

In terms of hours, the thermal admittance time lead φ is

$$\lambda = \frac{24}{2\pi} \times 0.63633 = 2.4036 \text{ hr } (2 \text{ hr } 26 \text{ min})$$

The peak heat flux *into* the wall occurs 2 hr 26 min before the peak environmental temperature.

11 Solar gain

11.1 Heat flow model

Solar radiation passing through windows will impinge on the inside surfaces of the walls. Some of the radiation will be absorbed and some will be reflected. The reflected radiation will impinge on other surfaces and, again, some will be absorbed and some will be reflected. Eventually, all of the solar radiation will be absorbed.

The absorbed radiation will heat up the walls. Some of the heat will pass through the walls to the outside and some will pass into the air adjacent to the walls by natural convection. Ref. [3] recommends a simple model to estimate the heat flux into the building and contributing to the net heat flux $q_{i,net}(t)$ [W m^{-2}].

We shall denote the radiative flux impinging on the wall by $q_{sg}(t)$ [W m^{-2}], the heat flux released to the air by $q_i(t)$ [W m^{-2}], and the heat flux passing through the wall by $q_o(t)$ [W m^{-2}], and we shall assume that the sizes of the fluxes $q_i(t)$, and $q_o(t)$ are inversely proportional the impedence to each flux. For $q_o(t)$ the impedence is equal to the impedence of the wall and boundary layers minus the impedence of the inner boundary layer, $Z-Z_{si}$. For $q_i(t)$ the impedence is due to the inner boundary layer, Z_{si}. Thus

$$\frac{q_o(t)}{q_i(t)} = \frac{q_{sg}(t) - q_i(t)}{q_i(t)} = \frac{Z_{si}}{Z - Z_{si}}$$

We can rearrange this equation as follows:

$$q_i(t) = \left(1 - \frac{Z_{si}}{Z}\right) q_{sg}(t)$$

The impedence is the reciprocal of the admittance. For the wall the admittance is Y_c if we ignore the fact that Y_c includes the inner boundary. For the inner boundary layer the impedence is simply the thermal resistance of the boundary layer $R_{si} = 1/h_i$. Thus,

$$q_i(t) = (1 - R_{si}Y_c)q_{sg}(t) \qquad (11.1)$$

11.2 Surface factor

The complex constant in (11.1) is denoted by F_c [ ]:

$$F_c = 1 - R_{si}Y_c \qquad (11.2)$$

where R_{si} [m^2 K W^{-1}] is the convective surface resistance of the inside surface.

The complex constant F_c can be written in the modulus-argument form:

$$F_c = |F_c| e^{j\mathrm{Arg}(F_c)} \quad (11.3)$$

where

$$|F_c| = \sqrt{\mathrm{Re}(F_c)^2 + \mathrm{Im}(F_c)^2} \quad (11.4)$$

is the modulus or amplitude of F_c and

$$\mathrm{Arg}(F_c) = \mathrm{atan}\left[\frac{\mathrm{Im}(F_c)}{\mathrm{Re}(F_c)}\right] \quad (11.5)$$

The terms $|F_c|$ and $\mathrm{Arg}(F_c)$ in (11.3) have special names. $|Fc|$ is known as the *surface factor* and is denoted by F. $\mathrm{Arg}(F_c)$ is known as the *surface factor time lag* and is denoted by ψ.

The parameter ψ is a time *lag* (the peak in $q_i(t)$ lags the peak in $q_{sg}(t)$) so ψ should always be negative. In tables of dynamic thermall parmeters, the surface factor time lag is given as a positive number, so we must place a negative sign in front of it in (11.3). Thus

$$F_c[\,] = Fe^{-j\psi} \quad (11.6)$$

The time-varying solar gain is

$$q_{sg}(t) = A_{sg}\sin(\omega t) = \mathrm{Im}\left(A_{sg}e^{j\omega t}\right) \quad (11.7)$$

where A_{sg} [W m^{-2}] is the amplitude of the solar-gain. The heat flux $q_i(t)$ [W m^{-2}] is

$$q_i(t) = \mathrm{Im}\left[F_c A_{sg} e^{j\omega t}\right] = \mathrm{Im}\left[Fe^{-j\psi} A_{sg} e^{j\omega t}\right] = A_{sg} F \sin(\omega t - \psi) \quad (11.8)$$

12 Example 4

For the composite wall with boundary layers in Ref. [1], calculate the surface factor F and the associated time lag ψ.

The complex constant F_c [ ] is defined by Eqn. (11.2):

$$F_c = 1 - R_{si} Y_c$$

where R_{si} [m^2 K W^{-1}] is the convective surface resistance of the inside surface.

For the brick wall the convection heat transfer coefficient of the inside surface h_i is 7.7 W m^{-2} K^{-1}, so

$$R_{si} = \frac{1}{h_i} = \frac{1}{7.7} = 0.12987 \text{ m}^2 \text{ K W}^{-1}$$

and the complex constant F_c is

$$F_c = 1 - 0.12987(0.73921 + j0.54617)$$

$$= 1 - 0.096001 - j0.070931$$

$$= 0.903999 - j0.070931$$

$$= 0.9068 e^{-j0.07830}$$

The surface factor F [ ] is the modulus of F_c:

$$F = |F_c| = 0.9068$$

The time lag ψ [hr] of F_c is 24 hr$/2\pi$ rad times the argument [rad] of F_c:

$$\psi = \frac{24}{2\pi} \times 0.07830 = 0.2991 \text{ hr (18 min)}$$

13 Example 5

The composite wall in Example 1 is subjected to the sinusoidal thermal conditions shown in Table 1. Use the dynamic thermal parameters calculated in Examples 2, 3 and 4 to determine the time varying net heat flux from the indoor side to the occupied space.

Table 1 Thermal conditions applied to the composite wall with boundary layers

Thermal condition	Mean	Amplitude	Time of +ve peak
Sol-air temperature	0°C	9°C	15:00 hr
Environmental temperature	0°C	4°C	12:00 hr
Solar gain	0 $\mathrm{Wm^{-2}}$	6 $\mathrm{Wm^{-2}}$	14:00 hr

Table 2 gives the dynamic thermal parameters calculated in Examples 2, 3 and 4.

Table 2 Dynamic thermal parameters for the composite wall with boundary layers

Decrement factor	Decrement factor time lag	Thermal admittance	Thermal admittance time lead	Surface factor	Surface factor time lag
f [ ]	ϕ [hr]	Y [W m^{-2} K^{-1}]	λ [hr]	F [ ]	φ [hr]
0.24170	8.7021	0.91909	2.4036	0.9068	0.2991

From Example 2, the steady thermal transmittance U is 0.58631 W m^{-2} K^{-1}.

<u>Decrement factor and decrement factor time lag</u>

The heat flux on the indoor side due to the sinusoidal variation in sol-air temperature is given by equation (7.7):

$$q_i(t) = A_{eo}U\mathrm{Im}\left[fe^{j[\omega t - 0.75\pi - \phi]}\right] = UfA_{eo}\sin(\omega t - 0.75\pi - \phi)$$

where A_{eo} is the amplitude of the sol-air temperature, U is the steady thermal transmittance of the wall, f is the decrement factor, and ϕ is the decrement factor time lag in radians. The peak in sol-air temperature occurs at 15:00 hr, so we have subtracted $(15{:}00 - 6{:}00)\times 2\pi/24$ from the argument of the sine function. The sol-air temperature variation is diurnal so the angular frequency ω is $2\pi/86400$ rad s^{-1}. The decrement factor time lag ϕ expressed in radians is

$$\phi = 8.7021 \times \frac{2\pi}{24} = 2.2782 \text{ rad}$$

Substituting the values of the parameters into (7.7) gives

$$q_i(t) = 0.58631 \times 0.24170 \times 9\sin\left(\frac{2\pi t}{86400} - 0.75\pi - 2.2782\right)$$

$$= 1.2754\sin(7.2722 \times 10^{-5}t - 0.75\pi - 2.2782) \qquad (13.1)$$

Thermal admittance and thermal admittance time lead

The heat flux on the indoor side due to the sinusoidal variation in environmental temperature is given by equation (9.3):

$$q_i(t) = A_{ei} Y \sin(\omega t - 0.5\pi + \varphi)$$

where A_{ei} is the amplitude of the environmental temperature, Y is the thermal admittance, and φ is the thermal admittance time lead in radians. The peak in environmental temperature occurs at 12:00 hr, so we have subtracted $(12{:}00 - 6{:}00) \times 2\pi/24$ from the argument of the sine function. The thermal admittance time lead φ expressed in radians is

$$\varphi = 2.4036 \times \frac{2\pi}{24} = 0.62926 \text{ rad}$$

Substituting the values of the parameters into (9.3) gives

$$q_i(t) = 4 \times 0.91909 \sin\left(\frac{2\pi t}{86400} - 0.5\pi + 0.62926\right)$$

$$= 3.6764 \sin(7.2722 \times 10^{-5} t - 0.5\pi + 0.62926) \qquad (13.2)$$

Note that this heat flux makes a *negative* contribution to the net heat flux $q_{i,net}(t)$ because the time lead is the time between the peak negative heat flux and the peak environmental temperature.

Surface factor and surface factor time lag

The heat flux on the indoor side due to the sinusoidal variation in the solar gain is given by equation (11.8):

$$q_i(t) = A_{sg} F \sin(\omega t - 2\pi/3 - \psi)$$

where A_{sg} [W m^{-2}] is the amplitude of the solar-gain, F is the surface factor, and ψ is the surface factor time lag. The peak in solar gain occurs at 14:00 hr, so we have subtracted $(14{:}00 - 6{:}00) \times 2\pi/24$ from the argument of the sine function. The surface factor time lag ψ expressed in radians is

$$\varphi = 0.2991 \times \frac{2\pi}{24} = 0.07830 \text{ rad}$$

Substituting the values of the parameters into (11.8) gives

$$q_i(t) = 6 \times 0.9068 \sin\left(\frac{2\pi t}{86400} - 2\pi/3 - 0.07830\right)$$

$$= 5.4408 \sin(7.2722 \times 10^{-5} t - 2\pi/3 - 0.07830) \qquad (13.3)$$

Net heat flux

The net heat flux $q_{i,net}(t)$ from the indoor side of the wall to the occupied space is obtained by summing (13.1), (13.2) and (13.3), remembering that (13.2) makes a negative contribution:

$$q_{i,net}(t) = 1.2754\sin(7.2722 \times 10^{-5}t - 0.75\pi - 2.2782)$$

$$-3.6764\sin(7.2722 \times 10^{-5}t - 0.5\pi + 0.62926)$$

$$+5.4408\sin(7.2722 \times 10^{-5}t - 2\pi/3 - 0.07830) \qquad (13.4)$$

Figure 5 shows the sinusoidal variation of the sol-air temperature and the environmental temperature. Figure 6 shows the heat fluxes generated on the indoor side of the wall by the sol-air temperature, the environmental temperature and the solar gain.

The positive peak in the sol-air temperature occurs at 15:00 hr. In Figure 6 we can see that the peak in the heat flux due to the sol-air temperature occurs 8 hr 42 min later.

The positive peak in the environmental temperature occurs at 12:00 hr. In Figure 6 we can see that the negative peak in the heat flux due to the environmental temperature occurs 2 hr 24 min earlier.

The positive peak in the solar gain occurs at 14:00 hr. The positive peak in the heat flux due to the solar gain occurs 18 min later.

Finally, we can see that the positive peak in the net heat flux occurs at 17 hr 48 min.

Figure 5 Sol-air and environmental temperatures

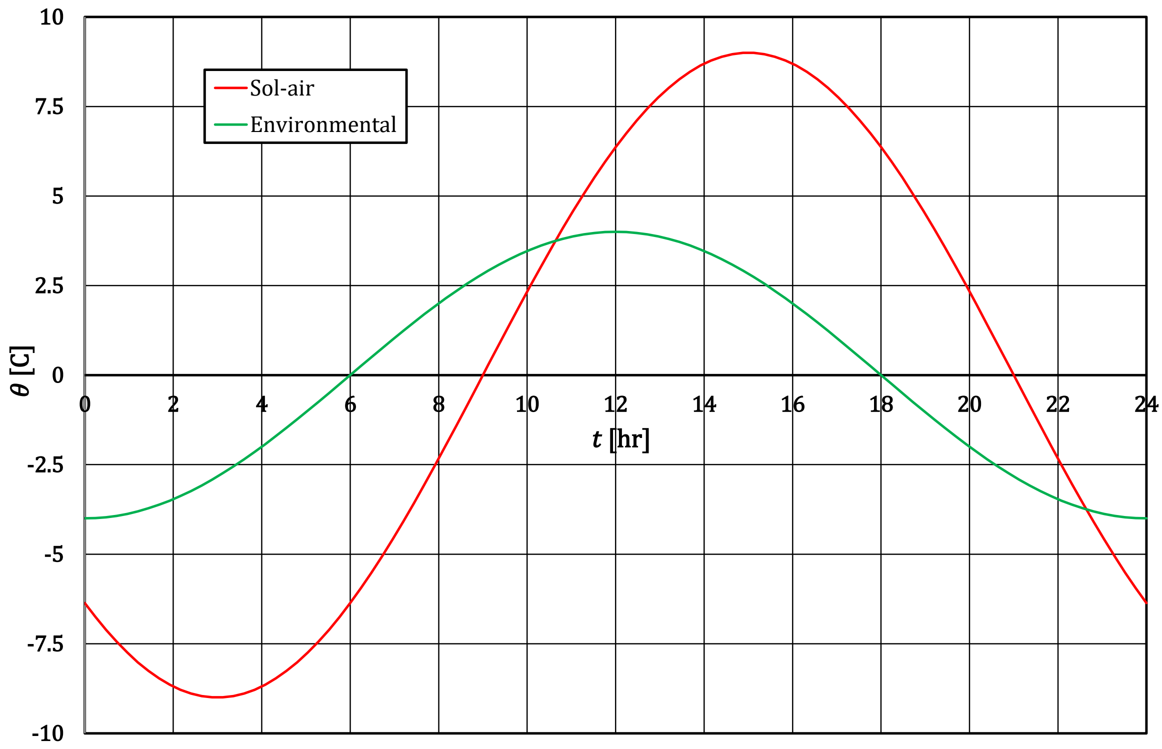
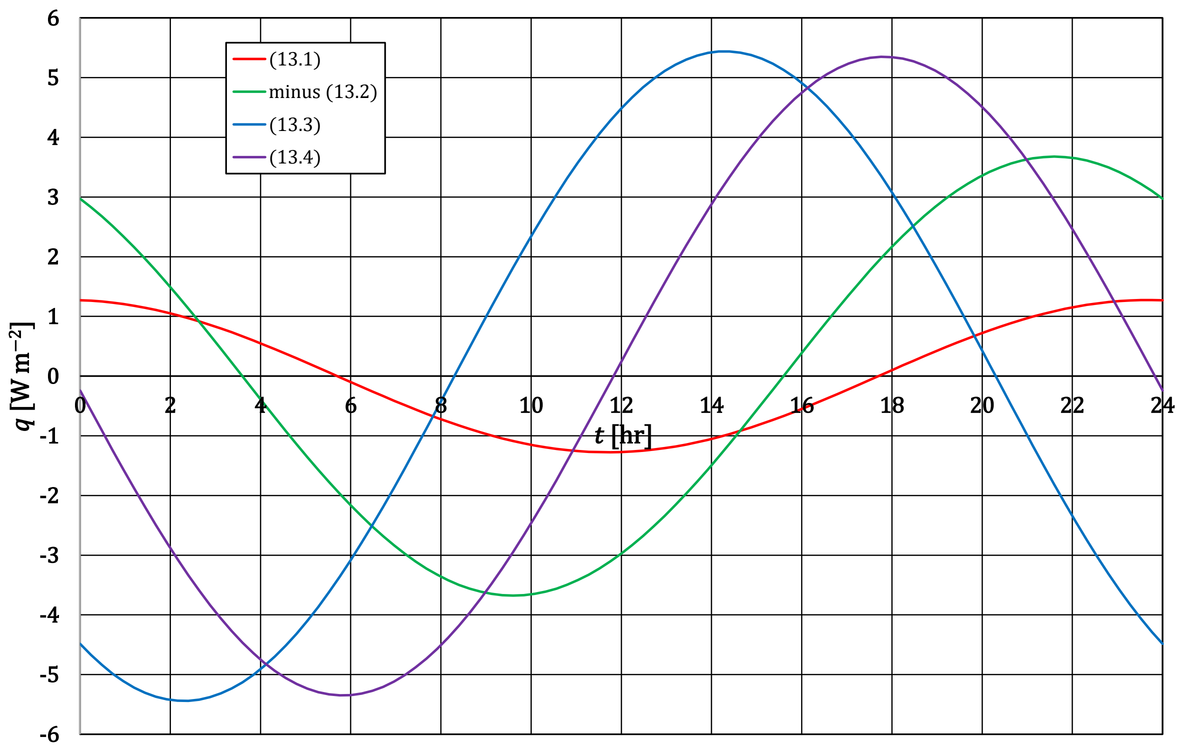


Figure 6 Heat fluxes from the indoor side of the wall to the occupied space



14 Heat capacity per unit area

The instantaneous heat flux through the inside surface of a wall $q_i(t)$ due to an oscillating environmental temperature is $\theta_{ei}(t)$ given by (3.3):

$$q_i(t) = \mathrm{Im}\left[A_{ei}\frac{Z_4}{Z_2}e^{j\omega t}\right] \quad (3.3)$$

The instantaneous heat flux through the outside surface $q_o(t)$ due to an oscillating environmental temperature is $\theta_{ei}(t)$ given by (3.4):

$$q_o(t) = \mathrm{Im}\left[\frac{A_{ei}}{Z_2}e^{j\omega t}\right] \quad (3.4)$$

Subtracting the complex constants in (3.3) and (3.4) from each other and taking the modulus provides a measure of the thermal storage capacity of the wall per unit area. The heat capacity per unit area χ [J K^{-1} m^{-2}] is defined by

$$\chi = \frac{P}{2\pi}\left|\frac{Z_4 - 1}{Z_2}\right| \quad (14.1)$$

where P [s] is the period of the temperature cycle.

15 Example 6

For the composite wall with boundary layers in Example 1, calculate the heat capacity per unit area χ.

The heat capacity per unit area χ is given by (14.1):

$$\chi = \frac{P}{2\pi}\left|\frac{Z_4 - 1}{Z_2}\right|$$

$$= \frac{86400}{2\pi}\left|\frac{-6.31991 + j1.45887 - 1}{4.58715 - j5.36282}\right|$$

$$= \frac{86400}{2\pi}\left|\frac{-7.31991 + j1.45887}{4.58715 - j5.36282}\right|$$

$$= \frac{86400}{2\pi}\left|\frac{(-7.31991 + j1.45887)(4.58715 + j5.36282)}{(4.58715 - j5.36282)(4.58715 + j5.36282)}\right|$$

$$= \frac{86400}{2\pi}\left|\frac{-33.5775 - j39.2554 + j6.69206 - 7.82366}{4.58715^2 + 5.36282^2}\right|$$

$$= \frac{86400}{2\pi}\left|\frac{-41.4012 - j32.5633}{49.8018}\right|$$

$$= \frac{86400}{2\pi}\left|-0.83132 - j0.65386\right|$$

$$= \frac{86400}{2\pi}\left|1.05765e^{-j0.90164}\right|$$

$$= \frac{86400}{2\pi} \times 1.05765 = 14544\ \mathrm{J\ K^{-1}\ m^{-2}}$$

$$= 14.544\ \mathrm{kJ\ K^{-1}\ m^{-2}}$$

16 Dynamic thermal parameters of typical wall constructions

This series of theory guides has explained the admittance method and shown how it is used to calculate some important dynamic thermal parameters of a planar composite wall. The decrement factor and its time lag, the thermal admittance and its time lead, and the surface factor and its time lag are required in the CIBSE cyclic model, which calculates the heat flows into and out of an occupied space during the course of a day. CIBSE have tabulated the values of the dynamic thermal parameters for different types of walls, roofs, ceilings and floors – see Ref. [2]. BSI also provide values – see Ref. [3]. If representative values of the parameters cannot be found in tables then they can be calculated using the methods in these theory guides.

17 References

1. K. N. Atkinson, *Admittance Method. 4. Convection and Radiation. Theory Guide*, Atkinson Science Limited, 2020.*
2. *CIBSE Guide A, Environmental design*, Chartered Institution of Building Services Engineers, 2015.
3. BS EN ISO 13786:2017, *Thermal performance of building components – Dynamic thermal characteristics – Calculation methods*, BSI Standards Limited, 2018.

* Download from https://atkinsonscience.co.uk/Downloads/Construction.aspx